Entry and exit sets in the dynamics of area preserving Hénon map

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Abstract

In this paper we study dynamical properties of the area preserving Hénon map, as a discrete version of open Hamiltonian systems, that can exhibit chaotic scattering. Exploiting its geometric properties we locate the exit and entry sets, i.e. regions through which any forward, respectively backward, unbounded orbit escapes to infinity. In order to get the boundaries of these sets we prove that the right branch of the unstable manifold of the hyperbolic fixed point is the graph of a function, which is the uniform limit of a sequence of functions whose graphs are arcs of the symmetry lines of the Hénon map, as a reversible map.

1   Introduction

The area preserving maps of the plane are discrete versions of open Hamiltonian systems that can exhibit chaotic scattering [7]. Their phase space is an open and unbounded set, that is the basic assumption of the Poincaré–recurrence theorem [1], concerning their dynamics, is violated. Hence it is natural to wonder which points of the plane are non–wandering, and through which regions do some orbits escape to infinity.

The simplest area preserving map of the plane is the quadratic Hénon map [8]:

$$(x, y) \mapsto (y + 1 - ax^2, x), \tag{1}$$

where a is a parameter. Despite its simplicity, the area preserving Hénon map exhibits a very complex dynamics, described in a long list of papers appeared during the thirty years from its definition. Even last five years important new results concerning the dynamic behaviour of this map were reported. In [12], for example, Meiss presents a theoretical support for the study of transport through the resonance zone associated to the hyperbolic fixed point of the Hénon map, as well as, for a numerical tool designed to compute the measure of the bounded orbits of the Hénon map. Homoclinic bifurcations for this map are discussed in [14], and the computation of its periodic orbits using anti–integrable limit in [13]. A method for computation of the self–rotation number for its orbits is given in [4].

In this paper we study the Hénon map from the point of view of open conservative systems, i.e. conservative systems whose phase space is unbounded. We prove the existence of an unbounded forward (backward) invariant set, called *exit set* (*entry set*) through which any forward (backward) unbounded orbit escapes to infinity. These sets are very useful in the study of transport properties of Hénon maps [11, 5], as well as of chaotic scattering in their dynamics [7, 10].

In order to locate these regions for the Hénon map, we study its geometric properties, and prove that the right branch of the unstable manifold of the hyperbolic fixed point is the graph of a function, which is the uniform limit of a sequence of functions whose graphs are arcs of the symmetry lines of the Hénon map, as a reversible map.

2   Regions of different dynamical behaviour for Hénon maps

Some dynamical properties of polynomial mappings with polynomial inverse are studied in [6]. These maps are called *Cremona maps* and they form a group G. Planar diffeomorphisms

$g_\delta \in G$ of the form

$$(x, y) \xmapsto{g_\delta} (y, p(y) - \delta x), \tag{2}$$

where p is a polynomial function of degree at least two, and $\delta \neq 0$ is the constant Jacobian determinant of the map g_δ, are called in [6] generalized Hénon transformations.

For $\delta = 1$ in (2) we get polynomial area preserving maps with polynomial inverse.

By [6] any Cremona map F of prime degree d is affinely conjugate either to a generalized Hénon map or to an elementary transformation $e(x, y) = (ax + p(y), by + c)$, $ab \neq 0$. In the first case the generalized Hénon map can have the polynomial p in the normal form $p(x) = \pm x^d +$ terms of degree $\leq d - 2$.

For our purposes it is more appropriate to study the class of area preserving generalized Hénon maps of the form g_1^{-1}, i.e. maps defined by:

$$(x, y) \xmapsto{H_d} (-y + p(x, \mu), x), \tag{3}$$

where $d = 2, 3$ is the degree of the polynomial family $p(x, \mu)$. Because in this case d is a prime number, a particular choice of the polynomial family $p(x, \mu)$ does not affect the generality of results. Namely, we study the quadratic Hénon map $H_2(x, y) = (-y + \mu x + x^2, x)$, and in a forthcoming paper the cubic Hénon maps $H_3^\pm(x, y) = (-y \pm x^3 + \mu x + \nu, x)$, as open systems.

In our study we exploit the reversibility property of the Hénon map. The area preserving maps (3) are reversible with respect to the involution $R : \mathbb{R}^2 \to \mathbb{R}^2$, $R(x, y) = (y, x)$, i.e. $H_d^{-1} = R \circ H_d \circ R$. Hence the map H_d factorizes as $H_d = I \circ R$, where $I(x, y) = (-x + p(y, \mu), y)$ is also an involution.

Recall some properties of the reversible systems we use in our approach (a survey on the dynamics of reversible systems is presented in [9]).

The relation $H_d^{-1} = R \circ H_d \circ R$ ensures that $H_d^{-n} = R \circ H_d^n \circ R$ also holds, for any $n \in \mathbb{Z}$. Hence H_d^n is R-reversible, too. Denote by I_n the involution defined by $I_n = H_d^n \circ R$, $n \in \mathbb{Z}$, and by Γ_n its fixed point sets, called n–symmetry line:

$$\Gamma_n = \{(x, y) \,|\, (H_d^n \circ R)(x, y) = (x, y)\} \tag{4}$$

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If R is a C^1–orientation reversing involution, then Γ_0 is nonempty, and moreover it is a C^1-curve in $\mathbb{R}^2$ having no self–intersections.

The symmetry lines Γ_k are transformed by H_d^n into other symmetry lines in the following way:

$$H_d^n(\Gamma_k) = \Gamma_{2n+k}, \ \forall \ n, k \in \mathbb{Z} \tag{5}$$

For the Hénon map (3), R and I are orientation reversing involutions and their fixed point sets are, respectively $\Gamma_0 : x = y$, $\Gamma_1 : x = p(y, \mu)/2$. An R–invariant periodic orbit of the map H_d is called symmetric orbit. The symmetric fixed points of the map H_d lie at the intersection of the two basic symmetry lines Γ_0 and Γ_1.

Hénon maps being discrete versions of open Hamiltonian systems, it is very important to locate the non–wandering set for such a map, and the region through which forward unbounded orbits escape to infinity. A point $z \in \mathbb{R}^2$ is non–wandering for the Hénon map H_d, if for any neighbourhood V of z, there exists an integer $n > 0$, such that $H_d(V) \cap V \neq \emptyset$. In other words, after some time $n > 0$, the orbit of a non–wandering point $z \in V$, comes back into the neighbourhood V.

The set of non-wandering points of the Hénon maps H_d, denoted $\Omega(H_d)$, is proved to be contained in a particular region of the phase space [6]. In this paper we locate the entry and the exit set for the quadratic Hénon map.

If a generalized Hénon map has no fixed points, then every orbit is unbounded [3]. In the case when (3) has fixed points, then there is a box B, such that any orbit outside it is either forward unbounded or backward unbounded or both. This box is:

$$B = \{(x, y) \mid |x| < M, |y| < M\}, \tag{6}$$

where M is the largest of the absolute values of the roots of the equation $|p(x, \mu)| - 2|x| = 0$. Hence bounded orbits lie inside the box.

The quadratic family $H_2(x, y) = (-y + \mu x + x^2, x)$ has two R–symmetric fixed points: $z_e = (0, 0)$ and $z_h = (2 - \mu, 2 - \mu)$. Denote $x_h := 2 - \mu$.

We study the maps corresponding to the parameter $\mu \in (-2, 2)$. In this range the origin is elliptic, while the second fixed point is hyperbolic. The value M in the definition of the box

B is, in this case $M = x_h$. The bounded orbits of the quadratic Hénon map, corresponding to $\mu \in (-2, 2)$, are periodic orbits, all orbits filling densely invariant circles surrounding the elliptic fixed point whose multipliers are $\lambda, \overline{\lambda} = e^{\pm 2\pi i\omega}$, with $\omega \neq 1/3, 1/4$, as well as orbits on the cantori (former invariant circles, which have broken down).

Beside the orbits outside the box B, it is possible to exist unbounded orbits which enter it, spend some time in B, and then are scattered to infinity. Chaotic scattering in the dynamics of area preserving maps of the plane was revealed for the quadratic map $f(x, y) = (\overline{x}, \overline{y})$ [10]:

$$
\begin{aligned}
\overline{x} &= [x - (x + y)^2/4]a \\
\overline{y} &= [y + (x + y)^2/4]a^{-1}
\end{aligned}
\tag{7}
$$

where $a > 1$ is a parameter.

Because any quadratic area preserving map of the plane, with nontrivial dynamics, is affinely conjugated to the Hénon map H_2, it results from the study of the map (7), that in some range of the parameter μ of the Hénon map can also exist a subset A in the box B, such that almost every point in A belongs to a trajectory that eventually leaves B (it is scattered to infinity). The stable set $S(A)$ (unstable set $U(A)$) is the set of points in A whose forward (backward) orbit stays in A. The set $C = S(A) \cap U(A)$ is H_2–invariant. If this set is a Cantor set which contains a dense orbit in C, of positive Lyapunov exponent, then C is called *chaotic saddle*. Hence the orbits of points on a chaotic saddle are also bounded. $A \setminus C$ is called *scattering region*. A trajectory entering the scattering region typically spends a finite amount of time near the chaotic saddle, and then exit it, as well as the box B, being scattered to infinity. The scattering process is called chaotic if the motion near the chaotic saddle is chaotic. Chaotic scattering is a manifestation of transient chaos in open Hamiltonian systems. In order to quantify the scattering process in the dynamics of area preserving maps one defines the *exit–time function*, and the *time–delay function* [7]. The exit–time function associates to an initial position in A the minimal time needed to reach the exit set, while the time–delay function associates to a point in the entry set the minimal time to reach the exit set. Each of these functions gives information on the time that the moving particle spends near the chaotic saddle, bouncing between its points. Because the points of

the chaotic saddle do not leave A, the exit–time function has a Cantor set of singularities.

As far as we know, the entry and exit sets of the polynomial area preserving maps, analyzed from the point of view of chaotic scattering, have not been located, and as a consequence the exit–time of a scattered particle z_0 was associated in a subjective way, namely as the the minimal time n such that the nth iterate of the map applied to z_0 is outside a large disk centered at the origin [10, 7]. Moreover the time–delay function could not be evaluated because the entry set was unknown.

Next we show that for the quadratic Hénon map we can determine analytically an exit set (entry set) of the system, i.e. an unbounded forward (backward) invariant set E (F), such that any forward (backward) unbounded orbit enter E (F), and as a consequence goes to infinity through this set.

The entry and exit sets are also useful in the study of transport through the resonance region, bounded by segments of stable and unstable manifolds of the hyperbolic fixed point of the Hénon map [11, 12, 5].

3 The exit and entry set of the quadratic Hénon map

In order to detect the exit and entry sets of the Hénon map we study its geometric properties. For simplicity we drop the index 2 from its name H_2.

Proposition 3.1 *The Hénon map $H(x, y) = (-y + \mu x + x^2, x)$, $\mu \in (-2, 2)$, has an unbounded forward invariant set $E_0 \subset \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, i.e. $H(E_0) \subset E_0$. Its boundary ∂E_0 is $\mathcal{L} \cup \Gamma_0^h$, where $\mathcal{L}$ is the curve parameterized by $(X(a), Y(a))$, $a \in (0, 1]$, with*

$$
\begin{aligned}
X(a) &= \frac{a^2 - a\mu + 1}{a} \\
Y(a) &= aX(a),
\end{aligned}
\tag{8}
$$

and $\Gamma_0^h = \{(x, y) \mid y = x, \ x \geq x_h\}$.

Proof

Let $L(a)$ be the infinite radius (semi–line) $y = a\,x$, starting from the elliptic fixed point $(0,0)$. Its image under H is an arc of parabola, $x = y^2 + (\mu - a)y$. Only the radii $L_1(a)$ defined by $x > 0$, $a > 0$, intersect their images $H(L_1(a))$ at the points $I(a) = \left(\dfrac{a^2 - a\mu + 1}{a^2}, \dfrac{a^2 - a\mu + 1}{a}\right)$ (Fig.1).

The point $I(a)$ is the H–image of the point $P(a) = \left(\dfrac{a^2 - a\mu + 1}{a}, a^2 - a\mu + 1\right)$. For any $\mu \in (-2, 2)$, $I(a)$, with $a > 0$, belongs to the set $S = \{(x, y) \,|\, x > 0, y > 0\}$, as well as the sub–arc of the parabola $x = y^2 + (\mu - a)y$ corresponding to $y > y_{I(a)}$. This suggests to look for a forward invariant set in S.

Any point $Q(x_0, a\,x_0) \in L_1(a)$, $0 < a \leq 1$, with $x_0 > x_{P(a)}$ is mapped by H to the point $(x_1, a'\,x_1) \in L_1(a')$ (see Fig.1), with

$$a'(x_0, a) = \frac{1}{x_0 + \mu - a} < a \tag{9}$$

Let us show that the set points

$$E_0 = \left\{(x, a\,x) \,\Big|\, x > \frac{a^2 - a\mu + 1}{a}, 0 < a \leq 1\right\} \tag{10}$$

is forward invariant.

For, take a point $(x_0, a\,x_0)$ in E_0. By (9), $\dfrac{a'^2 - a'\mu + 1}{a'} = \dfrac{1}{x_0 + \mu - a} + x_0 - a$. From $a' < a$, we have $\dfrac{a'^2 - a'\mu + 1}{a'} < x_0$. For $0 < a \leq 1$, $\dfrac{a^2 - a\mu + 1}{a} > 1 - \mu + a$. Hence $x_0 > 1 - \mu + a$, i.e. $x_0 + \mu - a > 1$, and thus $x_1 > x_0$ or equivalently $x_1 > \dfrac{a'^2 - a'\mu + 1}{a'}$. This means that $(x_1, a'\,x_1) = H(x_0, a\,x_0) \in E_0$, for any $(x_0, a\,x_0) \in E_0$. $\square$

Remark 3.1 *In canonical polar coordinates, a point $(r_0, \theta_0) \in E_0$ is mapped by H to a point (r_1, θ_1), with $\theta_1 < \theta_0$. Any point $(r_0, \theta_0) \in (0, \infty) \times [\pi/2, 2\pi]$, is mapped to a point (r_1, θ_1), where θ_1 is obtained from θ_0 by a positive translation (rotation) on the circle.*

Proposition 3.2 *The forward orbit $(x_n, y_n) = H^n(x_0, y_0)$, $n \in \mathbb{N}$, of a point $(x_0, y_0) \in E_0$ has the property that both sequences $(x_n), (y_n)$ are increasing and unbounded.*

Proof

Projection onto the first factor of the set E_0 is (x_h, ∞). Hence $x_0 > x_h$. From the proof of the Proposition 3.1 it results that the sequence (x_n) is increasing. Moreover

$$x_{n+1} + x_{n-1} = p(x_n), \tag{11}$$

with $p(x) = \mu x + x^2$. The equation (11) is the Hénon map expressed as the second order difference equation. The sequence (x_n) is unbounded, because otherwise it converges to a point x^* satisfying, by (11), the fixed point equation associated to the Hénon map, $p(x^*) = 2x^*$, i.e. $x^* \notin \mathrm{pr}_1(E_0)$.

In a similar way one proves that the sequence (y_n) is increasing and unbounded (y_n satisfies the same second order difference equation). $\square$

As a consequence, in canonical polar coordinates, the orbit (r_n, θ_n), $n \in \mathbb{N}$, of a point $(r_0, \theta_0) \in E_0$, has $r_n = (x_n^2 + y_n^2)/2 \to \infty$, i.e. the orbit goes to infinity.

Let $\lambda^u > 1$ be the expanding multiplier of the hyperbolic fixed point z_h, and $v = (\lambda^u, 1)$ a corresponding eigenvector. The branch of the unstable manifold $W^u(z_h)$ having the direction and the sense of v for the tangent semi-line l at z_h is denoted by $W^u_+(z_h)$, and called the right branch of the unstable manifold.

Proposition 3.3 *The right branch $W^u_+(z_h)$ of the unstable manifold of the hyperbolic fixed point z_h is included in the set E_0.*

Proof

A simple computation shows that the segment $l = \{z \in \mathbb{R}^2 | z = z_h + tv, 0 < t \leq 1\}$ of the tangent at z_h, to the unstable manifold, is included in the set E_0. Hence, there is a $\delta > 0$ such that an arc of the local unstable manifold $W^u_{loc}(z_h) = \{z \in \mathbb{R}^2 | H^{-n}(z) \in B_\delta(z_h), \forall n \geq 0\}$ is included in E_0 ($B_\delta(z_h)$ is the δ–ball centered at z_h). Denote by L^u this arc. The right branch of unstable manifold is then $W^u_+(z_h) = \bigcup_{n \geq 0} H^n(L^u)$, and by the forward invariance of E_0, it is included in this set. $\square$

The Hénon map H being reversible, its inverse is conjugated with H by the involution $R(x, y) = (y, x)$. Hence the set $F_0 = R(E_0)$ is a backward invariant subset, i.e. $H^{-1}(F_0) \subset$

F_0. Its boundary is $\partial F_0 = \mathcal{L}' \cup \Gamma_0^h$, where $\mathcal{L}' = R(\mathcal{L})$. Furthermore, the right branch, $W_+^s(z_h) = R(W_+^u(z_h))$, of the stable manifold $W^s(z_h)$, is included in the set F_0.

Next we determine the maximal forward, respectively backward, invariant set in $S = \{(x, y) \mid x > 0, y > 0\}$. For, we exploit the fact that $\Gamma_0^h = E_0 \cap F_0$, i.e. orbits starting from points $z_0 = (x_0, x_0)$, with $x_0 > x_h$, are unbounded in both directions. Moreover, $\Gamma_{-2n}^h := H^{-n}(\Gamma_0^h) \subset F_0$, and $\Gamma_{2n}^h := H^n(\Gamma_0^h) \subset E_0$, $\forall\, n \in \mathbb{N}$.

Because $\Gamma_0^h \subset E_0$ it follows that the sequence of sets (E_n), $n \in \mathbb{N}$, with $\partial E_n = \mathcal{L} \cup \Gamma_{-2n}^h$, is a sequence of forward H–invariant sets. We prove that it is an increasing sequence and its limit has as a partial boundary the right branch, $W_+^s(z_h)$, of the stable manifold of the hyperbolic fixed point z_h.

Proposition 3.4 *The right branch, $W_+^s(z_h)$, of the stable manifold is the graph of an increasing C^1–function $\varphi : [x_h, \infty) \to [x_h, \infty)$, which is the uniform limit of a sequence of increasing C^1–functions $\varphi_n : [x_h, \infty) \to [x_h, \infty)$, $n \in \mathbb{N}$, and:*
i) $\varphi_n(x_h) = x_h$, $\forall\, n \in \mathbb{N}$;
ii) $\varphi_n(x) > \varphi_m(x)$, $\forall\, n > m$, and $x > x_h$;
iii) the graph of the function φ_n is the partial boundary Γ_{-2n}^h of the forward invariant set E_n.

Proof

Adapting a result from [2] to the case of area preserving Hénon map, we have that the right branch $W_+^u(z_h)$ of the unstable manifold is the graph of an increasing C^1–function $g : [x_h, \infty) \to (x_h, \infty)$. The right branch of stable manifold $W_+^s(z_h) = R(W_+^u(z_h))$ being the symmetric of $W_+^u(z_h)$ with respect to the line $y = x$, is the graph of the function $\varphi = g^{-1}$, and this function is also increasing.

The partial boundary Γ_0^h of the set E_0 is the graph of the function $\varphi_0 = \varphi_0^{-1}$, $\varphi_0(x) = x$, $x \geq x_h$. Its image under H is the set of points $H(x, \varphi_0^{-1}(x)) = (-\varphi_0^{-1}(x) + p(x), x)$, with $x \geq x_h$, and $p(x) = \mu x + x^2$. Denote by $\varphi_1(x) := p(x) - \varphi_0^{-1}(x)$. Obviously, $\varphi_1(x_h) = x_h$, and $\varphi_0(x) < \varphi_1(x)$, $\forall\, x > x_h$. The derivative of φ_1 has the property:

$$\varphi_1'(x) > 1, \ \forall\, x \geq x_h \tag{12}$$

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Hence it is an increasing, that is invertible function, and the graph of its inverse is Γ_2^h, while the graph of φ_1 is $\Gamma_{-2}^h = R(\Gamma_2^h)$.

Γ_4^h is thus the H–image of the graph of the function φ_1^{-1}, i.e. the set of points $H(x, \varphi_1^{-1}(x))$ $= (p(x) - \varphi_1^{-1}(x), x)$, $x \geq x_h$. Denote by φ_2 the function defined by $\varphi_2(x) := p(x) - \varphi_1^{-1}(x)$. $\varphi_2(x_h) = x_h$. Because $\varphi_0(x) < \varphi_1(x)$, $\forall\, x > x_h$, is equivalent to $\varphi_1^{-1}(x) < \varphi_0^{-1}(x)$, $\forall\, x > x_h$, we get from the definition of the functions φ_i, $i = 1, 2$, that $\varphi_2(x) - \varphi_1(x) = \varphi_0^{-1} - \varphi_1^{-1} > 0$. The property (12) and the fact $p'(x) > 2$, for any $x \geq x_h$, ensures that $\varphi_2'(x) > 1$, $\forall x \geq x_h$. Hence the graph of φ_2^{-1} is Γ_4^h, and the graph of φ_2 is Γ_{-4}^h. By induction, we get that the partial boundary Γ_{-2n}^h of the set E_n, is the graph of an increasing C^1-function $\varphi_n : [x_h, \infty) \to [x_h, \infty)$, defined by:

$$\varphi_n(x) := p(x) - \varphi_{n-1}^{-1}, \ \forall\, x \geq x_h, \tag{13}$$

and it fulfills the conditions: $\varphi_n(x_h) = x_h$, $\varphi_n(x) > \varphi_{n-1}(x)$, $\forall\, x > x_h$, and $\varphi_n'(x) \geq 1$.

No arc Γ_{-2k}^h can intersect the stable manifold $W^s(z_h)$ in F_0. To prove this property, suppose that there exists a point $z \in W_+^s(z_h) \cap \Gamma_{-2k}^h$, $z \neq z_h$. Hence $\lim_{n \to \infty} H^n(z) = z_h$. Because $\Gamma_{-2k}^h \subset \Gamma_{-2k}$, by (4) we have $(H^{-2k} \circ R)(z) = z$. At the same time, $Rz \in W^u(z_h)$, and $\lim_{n \to \infty} H^{-n}(Rz) = z_h$ or equivalently $\lim_{n \to \infty} H^{-n+2k}(H^{-2k}R(z)) = z_h$. Hence the orbit of z is a homoclinic orbit to $z_h \notin F_0$, which contradicts the fact that z is a point in the backward invariant set F_0.

Because $W_+^s(z_h)$ cannot intersect any arc of symmetry line Γ_{-2n} in F_0, we have that $\varphi_n(x) < \varphi(x)$, for any $x > x_h$. Hence the sequence of functions (φ_n) converges uniformly to an increasing C^1–function $\psi : [x_h, \infty) \to [x_h, \infty)$. By (13) this function has the property that $p(x) - \psi(x) = \psi^{-1}(x)$. Hence a point $(x_0, \psi(x_0))$ of its graph is mapped by the Hénon map to $H(x_0, \psi(x_0)) = (p(x_0) - \psi(x_0), x_0) = (\psi^{-1}(x_0), x_0)$. Denoting $x_1 = \psi^{-1}(x_0)$, we have $H(x_0, \psi(x_0)) = (x_1, \psi(x_1))$, i.e. the graph of the function ψ is forward invariant under the action of the map H. If $(x_n, \psi(x_n)) = H^n(x_0, \psi(x_0))$, then $(x_{n-1}, \psi(x_{n-1})) = H^{-1}(x_n, \psi(x_n))$. By Proposition 3.2 and the reversibility of the system we get that $x_{n-1} > x_n$, $\forall\, n > 0$, and $x_n \to x_h$. Hence $H^n(x_0, \psi(x_0)) \to (x_h, x_h)$, as $n \to \infty$, that is, the graph of ψ is the right branch of the stable manifold of the fixed point z_h, i.e. $\psi = \varphi$. $\square$

Corollary 3.1 *The sequence of forward invariant sets E_n having the boundary $\partial E_n = \mathcal{L} \cup \Gamma^h_{-2n}$ is an increasing sequence and its limit $E = \cup_{n \geq 0} E_n$ is forward invariant, too, and has the boundary $\partial E = \mathcal{L} \cup W^s_+(z_s)$.*

The set $F = R(E)$ is backward invariant, and $\partial F = \mathcal{L}' \cup W^u_+(z_s)$.

The set of points $G = E \cap F = \{(x,y) \,|\, x > x_h, \varphi^{-1}(x) < y < \varphi(x)\}$ is H–invariant, i.e. $H(G) = G$.

Because the orbit of a point in G is unbounded in both directions, the exit and entry sets of dynamical interest for transport processes, as well as for chaotic scattering are $E' = E \setminus G$, respectively $F' = F \setminus G$. The boundary for E' is $\partial E' = \mathcal{L} \cup W^u_+(z_h)$, while for F' is $\partial F' = \mathcal{L}' \cup W^s_+(z_h)$.

The position of these sets in the phase space of the map is illustrated in Fig.2a, and the position of the invariant manifolds of the hyperbolic fixed point, with respect to these regions is shown in Fig.2b.

By straightforward computation we get that the preimage of the curve $\mathcal{L}$, as well as the image of $\mathcal{L}'$, intersects the box B. Hence there exist in the box B, points z, such that $t^+(z) = \min\{n \geq 0 \,|\, H^n(z) \in E'\} = 1$, respectively points z' with $t^-(z') = \min\{n \geq 0 \,|\, H^{-n}(z) \in F'\} = 1$. This means that indeed E', F' are exit, respectively entry, set for the Hénon map, as a map having a scattering region.

Knowing these sets, a typical scattering experiment consists in launching particles $z \in F'$ and observing their trajectory. If such a trajectory enters the box B and then exits and reaches E', the time–delay is the time it takes to a particle to go from the launching position until it reaches the exit set E. To a particle $z \in F$, which do not enter B, one associates the time 0. The existence of particles in F' that do not enter B is ensured by the fact that $H(\mathcal{L}')$ intersect the complementary set of B, not only the set B.

In Fig.3 is illustrated the time–delay function of points in the entry set F' belonging to the segment $y = 3.61$, $x \in [2.66, 2.74]$.

Note that the points of the curve $\mathcal{L}$ are mapped to points in E'. Namely, the points on $\mathcal{L}$ are the points $P(a)$ in the Proposition 3.1, and they are applied to points $I(a) \in E'$, i.e. to

points on the same semiline of slope a (see Fig.1). Moreover, a simple analytical computation shows that points on the semi–line $y = 0$, $x > x_h$, as well as points in the complement of the box B, lying between this semi–line and the curve $\mathcal{L}$, are applied in few iterations of the map into the set E'. Taking also into account the Remark 3.1, we conclude that any point from the exterior of the set B, having a forward unbounded orbit, reaches the exit set E, in a small number of iterations.

4   Conclusions

Studying geometric properties of the quadratic area preserving Hénon map, we have located in the phase space its exit and entry sets. The knowledge of these sets allows a more rigorous study of transport properties, as well as of chaotic scattering processes in the dynamics of this map. It appears that unlike classical Hamiltonian systems, exhibiting chaotic scattering, in the dynamics of the Hénon map, the particles cannot enter the scatterer from any direction of the phase space, and cannot exit from the system in any direction. The entry (exit) set is located in a neighbourhood of a branch of stable (unstable) manifold of the hyperbolic fixed point of the map.

In a forthcoming paper we determine these sets for generalized Hénon maps of degree three, which have either an elliptic fixed point and two hyperbolic fixed points, or a single elliptic point, but a two periodic hyperbolic orbit.

5   Acknowledgements

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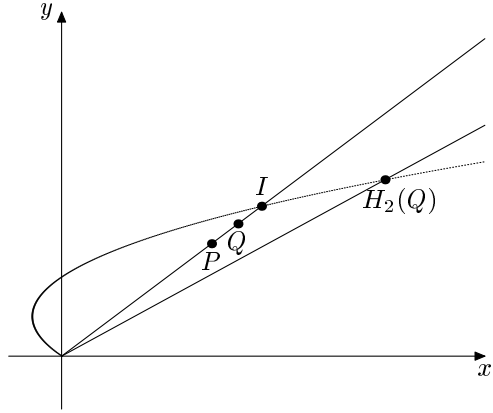
Fig. 1: The action of the map H on a semi–line $L_1(a)$, $a \leq 1$.
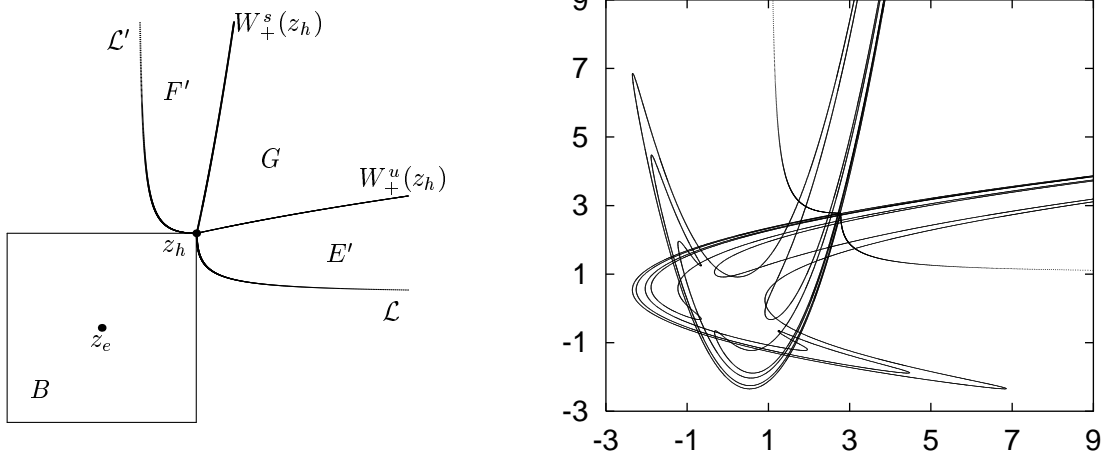

Fig. 2: a) Regions of different dynamical behaviour of the Hénon map, corresponding to $\mu = -0.78$; b) The invariant manifolds of the hyperbolic fixed point. Within the sets E' (F') one can see foldings of the left branch of the unstable (stable) manifold of the point z_h.
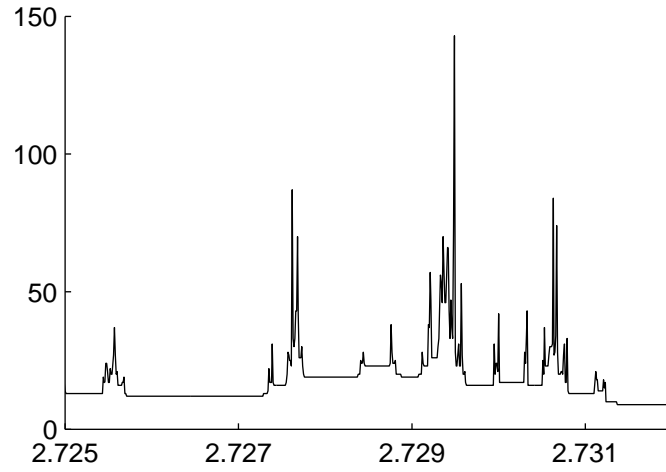
15

Fig. 3: The time–delay function defined on the segment $y = 3.61$, $x \in [2.66, 2.74]$ included in the entry set F' of the Hénon map corresponding to $\mu = -0.75$.